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BOUNDARY ELEMENT FORMULATIONS FOR LARGE STRAIN-LARGE DEFORMATION PROBLEMS OF VISCOPLASTICITY

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Abstract—Boundary element formulations with numerical solution strategies, for viscoplasticity problems in the presence of large strains and large deformations, are presented in this paper.

INTRODUCTION

In many technological applications, a metallic body is subjected to finite strains and finite deformations. The commonest examples are metal forming processes such as extrusion, rolling, sheet metal forming, etc. The components of elastic strain, in these examples, are generally limited to about 10^{-3} because the elastic moduli of metals are typically about three orders of magnitude larger than the yield stress. Thus, the nonelastic strain components, which can be of the order of unity, greatly dominate the elastic strains.

A considerable body of literature exists where the rigid-plastic material model has been used to determine stresses and deformations in metallic components undergoing large strains and deformations. Such a material model neglects the elastic strain components altogether. This approach suffers from the disadvantage that stresses can only be calculated in the plastically deforming region which, in case of say extrusion, occupies only a small portion of the workpiece. Further, residual stresses in a processed workpiece cannot be determined from a rigid-plastic material model. Knowledge of residual stresses is of crucial importance since these stresses can vastly influence the life of the product.

The next step in material modelling is the employment of an elastic-time-independent plastic constitutive model. Such a model has been employed in recent years by several researchers and numerical results for sample problems have been obtained by the finite element method (FEM). A few important publications in this subject area are listed as Refs. [1-3]. An elastic-plastic model overcomes many of the shortcomings of the rigid-plastic model. Deformation of many metals, however, is known to be rate and time-dependent, even at room temperature. Thus, the use of an appropriate elastic-viscoplastic model appears more realistic than that of a rate independent plastic theory [4].

Once the material constitutive model has been decided upon, attention must be focussed on the mathematical technique that is used to solve boundary value problems. The theory of characteristics of partial differential equations has been applied in many analyses using the rigid-plastic model. More recent research with the elastic-plastic model has, almost exclusively, employed the finite element method.

The boundary element method (BEM—also called the boundary integral equation method) is another powerful general purpose method. Significant advances in the applications of the method to nonlinear problems have been made in the past decade. The reader is referred to some recent publications [5-7] in order to get an overview of the state-of-the-art of this subject.

The nonlinear problems in solid mechanics, that have been solved to date by the BEM, are problems with small strains and deformations, with the nonlinearities arising from the material model. The book by Mukherjee [7], for example, presents solutions to many time-dependent inelastic deformation problems including planar and axisymmetric deformation, torsion of shafts and bending of plates. The purpose of this paper is to explore the possibility of using the BEM to solve problems where both material and geometric nonlinearities (large strains and deformations) are present.

The paper begins with a discussion of elastic-viscoplastic constitutive models suitable for the description of material behavior under conditions of large strain and large deformation. This is followed by a discussion of a boundary element method formulation and numerical results for some simple problems.

CONSTITUTIVE MODELS

Kinematics

A three-dimensional body is considered in this paper. Referring to a set of spatially fixed rectangular cartesian coordinates, a material particle in the body in a reference configuration is assumed to have coordinates X_i. The same particle has coordinates x_i in the current configuration. The range of indices in this paper is 1, 2, 3.

The displacement vector u_i is defined by the equation

$$x_i = X_i + u_i. \tag{1}$$

The velocity of this material point during deformation is denoted by v_i. The deformation rate d_{ij} is the symmetric part of the velocity gradient

$$d_{ij} = (1/2)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2}$$

whereas the spin rate ω_{ij} is the anti-symmetric part

$$\omega_{ij} = (1/2)\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{3}$$

Small strain-small deformation

The constitutive equations under conditions of small strain and small deformation are first reviewed before those for large strain and large deformation are presented. In this case, it is assumed that the total strain rate $\dot{\epsilon}_{ij}$ (where $\epsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$ and a superimposed dot denotes the partial derivative with respect to time) can be linearly decomposed into an elastic strain rate $\dot{\epsilon}_{ij}^{(e)}$ and a nonelastic strain rate $\dot{\epsilon}_{ij}^{(n)}$. Thus

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(e)} + \dot{\epsilon}_{ij}^{(n)}. \tag{4}$$

The elastic strain rate is related to the rate of the Cauchy stress σ_{ij} through Hooke's law

$$\dot{\sigma}_{ij} = \lambda\dot{\epsilon}_{kk}^{(e)}\,\delta_{ij} + 2\mu\dot{\epsilon}_{ij}^{(e)} \tag{5}$$

where μ and λ are the usual Lamé constants and δ_{ij} is the Kronecker delta. The material is assumed to be homogeneous and isotropic.

The nonelastic strain rates must now be determined from an inelastic constitutive model. A possibility is the use of a viscoplastic constitutive model with state variables q_{ij}. Thus, one can assume, for example,

$$\dot{\epsilon}_{ij}^{(n)} = f_{ij}\quad(\sigma_{ij}, q_{ij}^{(k)}) \tag{6}$$

$$\dot{q}_{ij}^{(k)} = g_{ij}\quad(\sigma_{ij}, q_{ij}^{(k)}) \tag{7}$$

$$\dot{\epsilon}_{kk}^{(n)} = 0 \tag{8}$$

which says that the history dependence of the nonelastic strain rate at any time can be represented by the current value of the stress and suitably chosen state variables q_{ij} which can be scalars or tensors. Suitable evolution laws must be prescribed for these state variables. Equation (8) states the experimentally observed fact that the nonelastic component of the deformation is incompressible. Equations (4)–(8) are valid for uniform temperature so that thermal strains and temperature effects are neglected. In the interest of brevity, further discussion of these equations is avoided here and the reader is referred to Chap. 2 of Ref. [7] for a more detailed discussion.

Large strain-large deformation

The analogous equations for the case of large strain and large deformation are presented next. First, it is assumed that the rate of deformation tensor can be linearly decomposed into an elastic and a nonlinear piece (see Fig. 1).

$$d_{ij} = d_{ij}^{(e)} + d_{ij}^{(n)}. \tag{9}$$

The next step relating $d_{ij}^{(e)}$ to stress rate is very important. According to Fung [8], a hypoelastic material is one in which the components of a (proper) stress rate are homogeneous linear functions of the components of the (proper) rate of deformation. One form of such a law, under conditions of material isotropy, can be written as

$$\overset{*}{\sigma}_{ij} = \lambda d_{kk}^{(e)}\delta_{ij} + 2\mu d_{ij}^{(e)}. \tag{10}$$

The rates used in eqn (10) must be objective. The corotational or Jaumann rate of the Cauchy stress is chosen here, so that

$$\overset{*}{\sigma}_{ij} = \dot{\sigma}_{ij} + \sigma_{ik}\omega_{kj} + \sigma_{jk}\omega_{ki} \tag{11}$$

where $\dot{\sigma}_{ij}$ is the material derivative

$$\dot{\sigma}_{ij} = \frac{\partial \sigma_{ij}}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_k} v_k. \tag{12}$$

The deformation gradient d_{ij} is objective. Equation (10) is also equivalent to the one used by Lee [2, 3].

The nonelastic constitutive equations analogous to eqns (6)–(8) are proposed as

$$d_{ij}^{(n)} = f_{ij}(\sigma_{ij}, q_{ij}^{(k)}) \tag{13}$$

$$\overset{*}{q}{}_{ij}^{(k)} = g_{ij}(\sigma_{ij}, q_{ij}^{(k)}) \tag{14}$$

$$d_{kk}^{(n)} = 0. \tag{15}$$

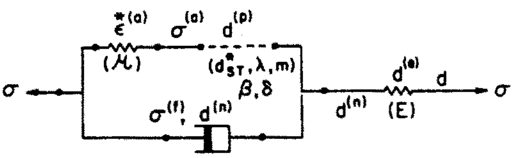

Fig. 1.

A BOUNDARY ELEMENT FORMULATION

A boundary element formulation for viscoplasticity problems in the presence of large strains and large deformations is presented in this section. The deformation of a body due to applied forces and displacements is analysed in real time t. The configuration of the body at time t is used as the reference configuration for the time step t to $t + \Delta t$. This is called an updated Lagrangian formulation [2–4] and leads to considerable simplification in the analysis.

The first assumption made here is that the deformation is nearly incompressible since the volume-preserving nonelastic deformation is expected to be much larger than the elastic deformation. Under this condition [4, 9]

$$\overset{*}{\tau}_{ij} \approx \overset{*}{\sigma}_{ij} \tag{16}$$

where τ_{ij} is the Kirchhoff stress tensor. The inner product of the Kirchhoff stress tensor with the unit normal in the reference configuration yields the traction in the reference configuration per unit reference area (see Table 1).

The Green–St. Venant strain E_{ij} is defined as

$$E_{ij} = (1/2)(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \tag{17}$$

(here a partial derivative with respect to X_i is denoted by $,i$) and the material derivative of the Green strain

$$\dot{E}_{ij} = d_{kl}x_{k,i}x_{l,j}. \tag{18}$$

Thus, in the updated Lagrangian approach

$$\dot{E}_{ij} = d_{ij} \tag{19}$$

since $x_{k,i}$ and $x_{l,j}$ reduce to δ_{ki} and δ_{lj} respectively.

Using eqn (16) (where equality is assumed) and eqn (19) and eqn (10) can be written as

$$\overset{*}{\tau}_{ij} = \lambda \dot{E}^{(e)}_{kk}\,\delta_{ij} + 2\mu \dot{E}^{(e)}_{ij}. \tag{20}$$

The BEM formulation is based on an appropriate form of Betti's reciprocal theorem. This is written as

$$\int_{B^0} \overset{*}{\tau}_{ij}\,\epsilon^{(R)}_{ij}\,\mathrm{d}V^0 = \int_{B^0} \sigma^{(R)}_{ij}\,\dot{E}^{(e)}_{ij}\,\mathrm{d}V^0 \tag{21}$$

where $\sigma^{(R)}_{ij}$, $\epsilon^{(R)}_{ij}$ are the reference stress and strain fields in an *elastic* body with the same Lamé parameters λ and μ undergoing *small strain* and *small deformation*. Since

$$\sigma^{(R)}_{ij} = \lambda\epsilon^{(R)}_{kk}\delta_{ij} + 2\mu\epsilon^{(R)}_{ij} \tag{22}$$

the integrands on either side of eqn (21) are identical in view of eqn (20). The integrals are to be evaluated over the domain B^0 of the reference configuration.

Using the equation $\epsilon^{(R)*}_{ij} = (1/2)(u^{(R)}_{i,j} + u^{(R)}_{j,i})$ and the divergence theorem, the left hand side of eqn (21) becomes

$$\text{L.H.S.} = \int_{\partial B^0} \overset{*}{\tau}_{ij} u^{(R)}_i\, n^0_j\,\mathrm{d}S^0 - \int_{B^0} \overset{*}{\tau}_{ij,j} u^{(R)}_i\,\mathrm{d}V^0. \tag{23}$$

Table 1. Stress measures

Stress Measure	Symbol	Symmetric?	Normal	Traction in - configuration	Traction per unit - surface area	Equation
Cauchy	σ_{ij}	yes	current	current	current	$n_j\sigma_{ji}ds = dT_i$
Lagrange	s_{ij}	no	reference	current	reference	$n^o_j s_{ji} ds^o = dT^{(L)}_i = dT_i$ (Lagrange rule)
Kirchhoff	τ_{ij}	yes	reference	reference	reference	$n^o_j \tau_{ji} ds^o = dT^{(k)}_i =$ $dT^o_i = \frac{\partial X_i}{\partial x_j} dT_j$ (Kirchhoff rule)

The rate of equilibrium equation in terms of the Lagrange stress tensor s_{ij} (see Table 1) using the updated Lagrangian approach is

$$\dot{s}_{ji,j} + \rho_0 \dot{F}_i^0 = 0 \tag{24}$$

where ρ_0 and $\dot{F}_i^0$ are the mass density and body force rate, respectively, in the reference configuration.

The relationship between the material derivative of the Lagrange stress in eqn (24) and the Jaumann rate of the Kirchhoff stress in eqn (23) is given by [4, 9]

$$\overset{*}{\tau}_{ij} = \dot{s}_{ij} + (\sigma_{ik} d_{jk} + \sigma_{kj} d_{ik}) - \sigma_{ik} v_{j,k}. \tag{25}$$

The above equation can be written as

$$\overset{*}{\tau}_{ij} = \dot{s}_{ji} + \overset{(*)}{G}_{jikl} v_{k,l} \tag{26}$$

where the tensor $\overset{(*)}{G}_{ijkl}$ is suitably defined by using eqns (25) and (2). In matrix form, one can write

$$\{\overset{*}{\tau}\} = \{\dot{s}\} + [\overset{(*)}{G}]\{L\} \tag{27}$$

where $L_{lk} = v_{k,l}$ and

$$\{L\}^T = [L_{11} L_{12} L_{13} L_{21} L_{22} L_{23} L_{31} L_{32} L_{33}].$$

The matrix $[\overset{(*)}{G}]$ is a function of components of the Cauchy stress σ_{ij} and is given in detail in Table 2.

The next two important steps in the derivation of the BEM equations are the use of eqns (24) and (26) to eliminate $\overset{*}{\tau}_{ij}$ and $\overset{*}{\tau}_{ij,j}$ from eqn (23) and the identification of the reference field with the usual Kelvin's solution for a point force in an infinite three-dimensional elastic solid undergoing infinitesimal deformation. Thus

$$u_i^{(R)} = U_{ij} e_j \tag{28}$$

$$\tau_i^{(R)} = T_{ij} e_j \tag{29}$$

$$\rho_0 F_i^{(R)} = \Delta(p, q)\, \delta_{ij} e_j \tag{30}$$

where U_{ij} and T_{ij} are the usual two-point kernels of small deformation elasticity that are given in many Refs. [7], p is the source point, q the field point, Δ the Dirac delta function, $F_i^{(R)}$ the point body force in the Kelvin solution and e_j is an appropriate component of one of a set of unit orthogonal base vectors. The reference fields satisfy the equilibrium equations

$$\sigma_{ij,j}^{(R)} = -\rho_0 F_i^{(R)} \text{ in } B^0 \tag{31}$$

and

$$\tau_i^{(R)} = n_j^0 \sigma_{ji}^{(R)} \text{ on } \partial B^0 \tag{32}$$

in the reference configuration.

It can be shown that by using eqns (24), (26), (28)–(30) and the divergence theorem, the expression on the right of eqn (23) becomes

$$\text{L.H.S.} = \int_{\partial B^0} U_{ij} \dot{s}_{ki} n_k^0 \, dS^0 + \int_{\partial B^0} U_{ij} \rho_0 \dot{F}_i^0 \, dV^0 + \int_{B^0} U_{ij,m} \overset{(*)}{G}_{mikl} v_{k,l} \, dV^0. \tag{33}$$

Table 2. The matrix $[\overset{*}{G}]$

$$
[\overset{*}{G}] = \left[\begin{array}{ccc|ccc|ccc}
\sigma_{11} & & & & \text{(SYMMETRIC)} & & & & \\
\sigma_{12} & -\frac{1}{2}(\sigma_{11}-\sigma_{22}) & & & & & & & \\
\sigma_{13} & \frac{1}{2}\sigma_{23} & -\frac{1}{2}(\sigma_{11}-\sigma_{33}) & & & & & & \\
0 & \frac{1}{2}(\sigma_{11}+\sigma_{22}) & \frac{1}{2}\sigma_{23} & -\frac{1}{2}(\sigma_{22}-\sigma_{11}) & & & & & \\
0 & 0 & 0 & \sigma_{12} & \sigma_{22} & & & & \\
0 & \frac{1}{2}\sigma_{13} & -\frac{1}{2}\sigma_{12} & \frac{1}{2}\sigma_{13} & \sigma_{23} & -\frac{1}{2}(\sigma_{22}-\sigma_{33}) & & & \\
0 & \frac{1}{2}\sigma_{23} & \frac{1}{2}(\sigma_{11}+\sigma_{33}) & \frac{1}{2}\sigma_{23} & 0 & \frac{1}{2}\sigma_{12} & -\frac{1}{2}(\sigma_{33}-\sigma_{11}) & & \\
0 & -\frac{1}{2}\sigma_{13} & \frac{1}{2}\sigma_{12} & \frac{1}{2}\sigma_{13} & 0 & \frac{1}{2}(\sigma_{22}+\sigma_{33}) & \frac{1}{2}\sigma_{12} & -\frac{1}{2}(\sigma_{33}-\sigma_{22}) & \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{13} & \sigma_{23} & \sigma_{33}
\end{array}\right]
$$

The expression on the right hand side of eqn (21) can be written as

$$\text{R.H.S.} = \int_{B^0} \sigma_{ij}^{(R)} [v_{i,j} - \dot{E}_{ij}^{(n)}] \, \mathrm{d}V^0. \tag{34}$$

Using eqns (22), (28)–(32), $\dot{E}_{kk}^{(n)} = 0$ and the divergence theorem, this expression becomes

$$\text{R.H.S.} = v_j + \int_{\partial B^0} T_{ij} v_i \, \mathrm{d}S^0 - \int_{B^0} 2GU_{ij,k} \dot{E}_{ik}^{(n)} \, \mathrm{d}V^0. \tag{35}$$

Finally, the complete boundary element formulation, from Betti's reciprocal theorem (eqn 21), is

$$\begin{aligned} v_j(p) = & \int_{\partial B^0} [U_{ij}(p, Q)\dot{\tau}_i(Q) - T_{ij}(p, Q) v_i(Q)] \, \mathrm{d}S_Q^0 \\ & + \int_{B^0} \rho_0 U_{ij}(p, q) \dot{F}_i^0(q) \, \mathrm{d}V_q^0 + \int_{B^0} 2GU_{ij,k}(p, q) \, d_{ik}^{(n)}(q) \, \mathrm{d}V_q^0 \\ & + \int_{B^0} U_{ij,m}(p, q) \overset{(*)}{G}_{mikl}(q) \, v_{k,l}(q) \, \mathrm{d}V_q^0 \end{aligned} \tag{36}$$

where $\dot{\tau}_i = \dot{s}_{ji} n_j^0$, eqn (34) has been employed and lower case letters p and q denote points inside B^0 and capital letters P and Q denote points on the boundary ∂B^0. A comma denotes differentiation at a field point q.

The first four terms on the right hand side of the above equation are analogous to terms in the BEM formulation for viscoplasticity with small strains and deformations [7]. The traction rate term $\dot{\tau}_i$ requires special care, as is explained in the next paragraph. The last domain integral comes about due to the presence of large strains and deformations. As can be seen, this term includes the unknown velocity field throughout the body. For plasticity problems, where $d_{ik}^{(n)}$ is a function of the stress rate (or, equivalently, a function of total strain rate) the presence of the last term does not present any added complication since, in any case, iterations are necessary within each load step. For viscoplasticity problems, $d_{ik}^{(n)}$ in which is an explicit function of the stress and state variables (but not of stress rate), iterations within each time step are now required because of the presence of the last domain integral. These iterations, however, can be easily taken care of, as demonstrated in the section describing numerical results to sample problems.

Using eqns (25) and (3), the traction rate $\dot{\tau}_i$ in eqn (36) can be written as

$$\dot{\tau}_j = \dot{s}_{ij} n_i = \dot{t}_j - \sigma_{jk} d_{ki} n_i + \omega_{jk} \sigma_{ki} n_i \tag{37}$$

where, using eqn (16)

$$\dot{t}_j = \overset{*}{\tau}_{ji} n_i = \overset{*}{\sigma}_{ji} n_i. \tag{38}$$

Eqn (37) is best interpreted in a local coordinate frame which translates with a material point on the boundary ∂B and rotates with the normal to the boundary at that point. In such a case, $\dot{t}_j$ becomes the components of the rate of the prescribed follower force, per unit deformed surface area, on the deforming boundary. The follower force moves with a boundary point and rotates with the normal to the boundary at that point. Such follower forces can usually be conveniently prescribed in many physical problems. The last two terms in eqn (37) must be included in a general analysis. Eqn (37) delivers the components of $\dot{\tau}_j$ in the local coordinate frame and these must be transformed back to the global frame for use in eqn (36). Chandra and Mukherjee [4] from example, describe how eqn (37) leads to a load correction matrix in an extrusion problem solved by the FEM.

It should be emphasized once again that the updated Lagrangian formulation is being used here. Thus for example, at any time "t", ∂B^0 the deformed boundary at that time.

Boundary displacement rates

A boundary integral equation can be obtained from eqn (36) in the usual way by taking the limit as $p \to P$. This gives

$$c_{ij}(P)v_j(P) = \int_{\partial B^0} [U_{ij}(P, Q)\dot{\tau}_i(Q) - T_{ij}(P, Q)v_i(Q)]\, \mathrm{d}S_Q^0 + \int_{B^0} \rho_0 U_{ij}(P, q)\dot{F}_i^0(q)\, \mathrm{d}V_q^0$$
$$+ \int_{B^0} 2GU_{ij,k}(P, q)\, d_{ik}^{(n)}(q)\, \mathrm{d}V_q^0 + \int_{B^0} U_{ij,m}(P, q)\overset{(*)}{G}_{mikl}(q)v_{k,l}(q)\, \mathrm{d}V_q^0. \qquad (39)$$

The coefficients of c_{ij}, in general, depend on the location of a boundary node and on the local geometry at P. If the boundary is locally smooth at P, $c_{ij} = (1/2)\partial_{ij}$. Otherwise, it can be evaluated indirectly for three-dimensional problems [7].

Stress rates

The first step towards the determination of stress rates is the differentiation of the velocity field from eqn (36) at an internal source point p. Thus,

$$v_{j,\bar{l}}(p) = \int_{\partial B^0} [U_{ij,\bar{l}}(p, Q)\dot{r}_i(Q) - T_{ij,\bar{l}}(p, Q)v_i(Q)]\, \mathrm{d}S_Q^0$$
$$+ \int_{B^0} \rho_0 U_{ij,\bar{l}}(p, q)\dot{F}_i^0(q)\, \mathrm{d}V_q^0 + \frac{\partial}{\partial x_{\bar{l}}}\int_{B^0} 2GU_{ij,k}(p, q)d_{ik}^{(n)}(q)\, \mathrm{d}V_q^0$$
$$+ \frac{\partial}{\partial x_{\bar{l}}}\int_{B^0} U_{ij,m}(p, q)\overset{(*)}{G}_{mikn}(q)v_{k,n}(q)\, \mathrm{d}V_q^0 \qquad (40)$$

where, $\bar{l} \equiv (\partial/\partial x_{\bar{l}})$ denotes differentiation with respect to a source point. The kernel $U_{ij,k}$ has a singularity of order r^{-2} (where r is the distance from a source to a field point) and cannot be directly differentiated under the integral sign. Several methods of evaluating such derivatives of the integrals are discussed in [7].

It is simple matter to obtain the stress rates $\overset{*}{\sigma}_{ij}$ and $\dot{\sigma}_{ij}$ once the velocity gradients are known throughout the body. These can be obtained by using eqns (2), (16), (12), (13), (3) and (14).

Boundary stress rates

The boundary stress rates cannot be conveniently obtained by taking the limit of eqn (40) as $p \to P$. Boundary stress rates as well as velocity gradients can, however, be accurately determined, by suing the method outlined in [7].

PLANAR PROBLEMS

Plane Strain ($v_3 = 0$)

The BEM equations for plane strain have exactly the same forms as the three-dimensional eqns (36), (39) and (40). The range of indices in this case is 1.2 and the appropriate expressions for the kernels U_{ij} and T_{ij} [7] and $\overset{(*)}{G}_{mikl}$ must be used.

Plane stress ($\sigma_{33} = 0$)

Once again, the range of indices for the plane stress equations is 1.2. The expressions for U_{ij} and T_{ij} for the plane stress problems have the same forms as those for plane strain with ν replaced by $\bar{\nu} = \nu/(1+\nu)$. One important difference is that the term

$$\int_{B^0} 2GU_{ij,k}(p, q)d_{ik}^{(n)}(q)\, \mathrm{d}V_q^0$$

in eqn (51) must be replaced by the term

$$\int_{B^0} \Sigma_{ikj}(p, q)d_{ik}^{(n)}(q)\, \mathrm{d}V_q^0.$$

The reason for this, as well as the appropriate expression for Σ_{ikj} is given in [7]. The appropriate change to Σ_{ikj} must, of course be carried through the plane stress versions of eqns (39) and (40). The last term in the plane stress version of eqn (36) still retains the kernel $U_{ij,m}$, i.e. it remains as

$$\int_{B^0} U_{ij,m}(p, q)\overset{(*)}{G}_{mikl}(q)v_{k,l}(q)\, dV_q^0$$

with, of course, the plane stress version of U_{ij} and $\overset{(*)}{G}_{mikl}$ being substituted for the three-dimensional ones.

NUMERICAL IMPLEMENTATION FOR PLANAR PROBLEMS

Numerical implementation of the BEM equations for plane strain problems is discussed in this section. It is understood that while the three-dimensional equations are referred to this section, their planar versions must be used in a numerical procedure.

The first step is the discretization of the two-dimensional body into boundary elements and internal cells. A discretized version of the boundary integral eqn (39) for the velocity is

$$\begin{aligned} c_{ij}(P_M)v_i(P_M) = {} & \sum_{N_s} \int_{\Delta s_N} U_{ij}(P_M, Q)\dot{\tau}_i(Q)\, dS_Q^0 - \sum_{N_s} \int_{\Delta s_N} T_{ij}(P_M, Q)v_i(Q)\, dS_Q^0 \\ & + \sum_{n_i} \int_{\Delta A_n} \rho_0 U_{ij}(P_M, q)\dot{F}_i^0(q)\, dV_q^0 + \sum_{n_i} \int_{\Delta A_n} 2GU_{ij,k}(P_M, q)d_{ik}^{(n)}(q)\, dV_q^0 \\ & + \sum_{n_i} \int_{\Delta A_n} U_{ij,m}(P_M, q)\overset{(*)}{G}_{mikl}(q)\, V_{k,l}(q)\, dV_q^0 \end{aligned} \tag{41}$$

where the boundary of the body ∂B^0 is divided into N_s boundary segments and the interior into n_i internal cells and $v_i(P_M)$ are the components of the velocities at a point P which coincides with node M.

Suitable shape functions must now be chosen for the variation of tractions and velocities on the surface elements Δc_N and the variation of the nonelastic strain rates and velocities over an internal cell ΔA_n. Integrals of kernels over elements on which they become singular must be obtained carefully. For planar problems with straight boundary elements and internal cells with straight boundaries, with fairly simple shape functions, it is possible to evaluate integrals of kernels analytically. This has been done in order to obtain the numerical results presented later in this chapter, and is recommended whenever possible.

A suitable strategy must be used for numerical modelling of possible jumps in normals or prescribed tractions across boundaries of boundary elements. A "zero length" element placed at a corner is convenient for this purpose [7].

Numerical discretization transforms eqn (41) into an algebraic system of the type

$$[A]\{v\} + [B]\{\dot{\tau}\} = \{b\} \tag{42}$$

where the coefficient matrices $[A]$ and $[B]$ contain integrals of the kernels and the shape functions and the vector $\{b\}$ contains the contributions of the various quantities from the three domain integrals.

Equations (36) and (40) for the velocity field and the velocity gradients at an internal point are discretized in similar fashion.

A solution strategy for viscoplasticity problems is described below. In essence, the strategy consists of marching forward in real time with suitable updating of the configuration of the body. The presence of velocity gradients in the boundary traction rates (see eqn 37) and in the last domain integral in eqn (39) requires iterations within each time step. The strategy is described below for the case of no prescribed body forces ($F_i = 0$), but these can be included without any difficulty. The strategy can be described as (see the flow chart, Fig. 2).

(a) The elasticity problem is first solved to obtain initial displacements. For this step, eqn (42) with $\{b\} = 0$ is solved for the unknown components of $\mathbf{u}$ and $\boldsymbol{\tau}$ in terms of the prescribed ones.

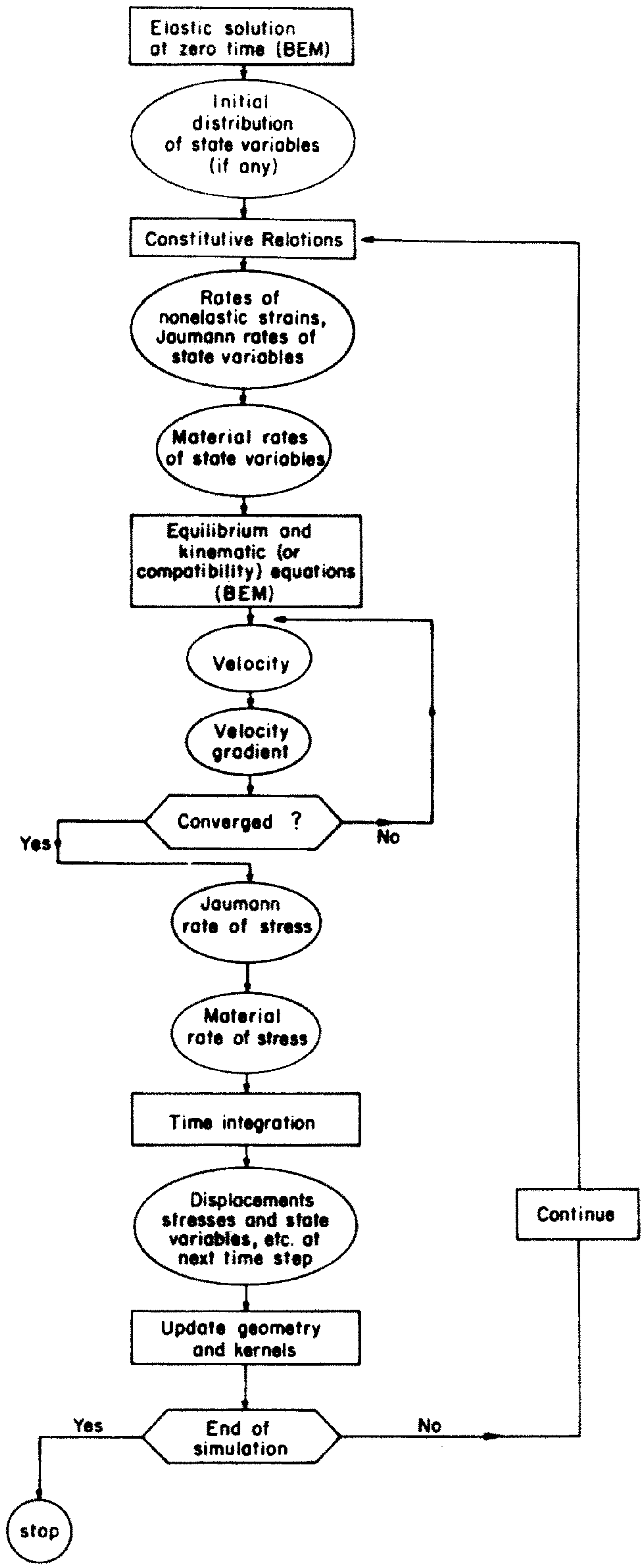
Elastic solution at zero time (BEM)
Initial distribution of state variables (if any)
Constitutive Relations
Rates of nonelastic strains, Jaumann rates of state variables
Material rates of state variables
Equilibrium and kinematic (or compatibility) equations (BEM)
Velocity
Velocity gradient
Converged ?
Yes
No
Jaumann rate of stress
Material rate of stress
Time integration
Displacements stresses and state variables, etc. at next time step
Update geometry and kernels
End of simulation
Yes
No
Continue
stop

Fig. 2.

(b) The initial values of the displacement gradients are obtained from a truncated version of eqn (40) with $d_{ik}^{(n)} = v_{k,n} = 0$ and **v** and $\dot{\tau}$ replaced by **u** and τ in the rest of the equation. The initial stresses are determined from the strains from Hooke's law. Also, initial rotations are determined from the displacement gradients.

(c) the tensor $d_{ik}^{(n)}$ at zero time is obtained from the constitutive eqns (16) and (18).

(d) The first approximation to $v_j(P)$ at $t = 0$ is calculated from eqn (54) with $v_{k,l}$ set to zero. It must be noted that the velocity gradients occur in one of the area integrals as well as in the expression for $\dot{\tau}_i$ through d_{ki} and ω_{jk} (see eqn 37).

(e) The first approximation to $v_{j,l}(p)$ at $t = 0$ is obtained from eqn (55) with all the velocity gradients in the integrals in the right hand side of eqn (40) set to zero. The first approximation to $v_{j,l}(P)$ is best obtained from a boundary stress rate algoritym [7].

(f) The first approximations to $v_{j,l}(p)$ and $v_{j,l}(P)$ are inserted into eqn (39) and the full equation is solved to determine a second approximation to $v_j(P)$. Next, this $v_j(P)$ and the first approximation to $v_{j,l}$ are used to the right hand side of eqn (40) to obtain the second approximation to $v_{j,l}(p)$.

(g) Step (f) is repeated as many times as necessary till convergence is achieved and $v_j(p)$ is determined at zero time.

(h) The quantities $v_j(p)$ (from eqn 51), $v_{j,l}(p)$ and then $\overset{*}{\sigma}_{ij}$ (from eqn 2) is calculated at zero time. Then, $\dot{\sigma}_{ij}$ is obtained from $\overset{*}{\sigma}_{ij}$ from eqns (3) and (14)).

(i) The relevant quantities (displacements, stresses, etc.) are calculated at time Δt from their values and rates at $t = 0$. The steps (c)–(h) are repeated to obtain the rates at $t = \Delta t$ and so on. The time histories of the various quantities are thus obtained by marching forward in time and suitable updating of the geometry and the kernels. Ideally, the geometry should be updated at each time step. In practice, updating is done as often as is necessary during the integration process. It should be noted that the time marching process must be carried out with the material derivatives of the tensors in order to obtain their integrated values in the global coordinate frame (original vector, basis $\underline{e}_1$, $\underline{e}_2$) as a function of **x** and time.

NUMERICAL RESULTS

As a first attempt to solve large strain large deformation problems of viscoplasticity by the BEM, uniaxial deformation problems (plane strain and plane stress) are considered. The two dimensional computer code, basedon the general equations for plane strain and plane stress problems given previously, has been used to generate these numerical results. For these problems, normals on straight boundaries do not rotate during deformation. The boundary conditions in these problems are either prescribed velocities, free surfaces or symmetry lines. Thus, all prescribed components of traction remain zero during deformation. This makes the computation less complicated, since appropriate components of $\dot{\tau}$ can be directly set to zero. The BEM formulation presented here is of course perfectly general and there is no difficulty in incorporating eqn (37) explicitly. Such problems are being considered by the authors in ongoing research at Cornell.

DISCRETIZATION, TIME INTEGRATION AND CONSTITUTIVE MODEL

The BEM program uses straight boundary elements and polygonal internal cells. The velocity and traction rate **v** and $\dot{\tau}$ are taken to be piecewise linear on the boundary elements and $d_{ij}^{(n)}$ and $v_{k,n}$ (in the last integral in eqn 40) are taken to be piecewise constant on the internal cells. The values of the boundary variables are assigned at nodes which lie at the intersections of boundary segments. Possible discontinuities in tractions are taken care of by placing a "zero length" element between nodes and assigning different values of traction at each of those nodes.

All integrations of kernels are carried out analytically. The last two terms in eqn (40) are evaluated by first performing the integration over an internal cell for an arbitrary source point p_m and then differentiating the integral at p_m. The procedures used here are quite similar to those described in Chap. 5 of [7]. The reader should consult this reference for further details.

The FEM formulation [4] used a piecewise quadratic description of velocities over triangular elements. Time integration for both the BEM and FEM programs is carried out by using an Euler type explicit method with automatic time-step control [10]. Hart's constitutive equations for large strain problems, described previously in [11] are used to model material behavior of 304 stainless steel at 400°C. The material parameters are available in Chap. 5 of [7].

Numerical results for sample problems

The first class of problems considered are those of an uniform slab (plane strain) or uniform plate (plane stress) held at one end and pulled at an uniform velocity at the other end. A quarter of the plate is modelled. For these simple problems, it is possible to obtain a direct solution with an imposed velocity field

$$\underline{v} = v_1(x_1)\underline{i} + v_2(x_2)\underline{j}$$

for plane strain and stepwise integration in time. A similar direct solution can be obtained for plane stress.

A comparison of stress–strain plots at an interval point are shown in Fig. 3 (plane strain) and Fig. 4 (plane stress). The interval point is chosen to lie at the center of the quarter plate. The results include the effect of material as well as geometrical nonlinearities. The solutions are seen to agree quite well up to a large amount (35%) of strain. The FEM as expected from a displacement formulation, is seen to predict somewhat stiffer behavior compared to the direct solution. The BEM predicts somewhat softer behavior compared to the direct solution.

The computer times for the BEM and FEM calculations, on an IBM 370/168 are shown in Table 3. The BEM times are seen to be considerably less than the FEM in both cases.

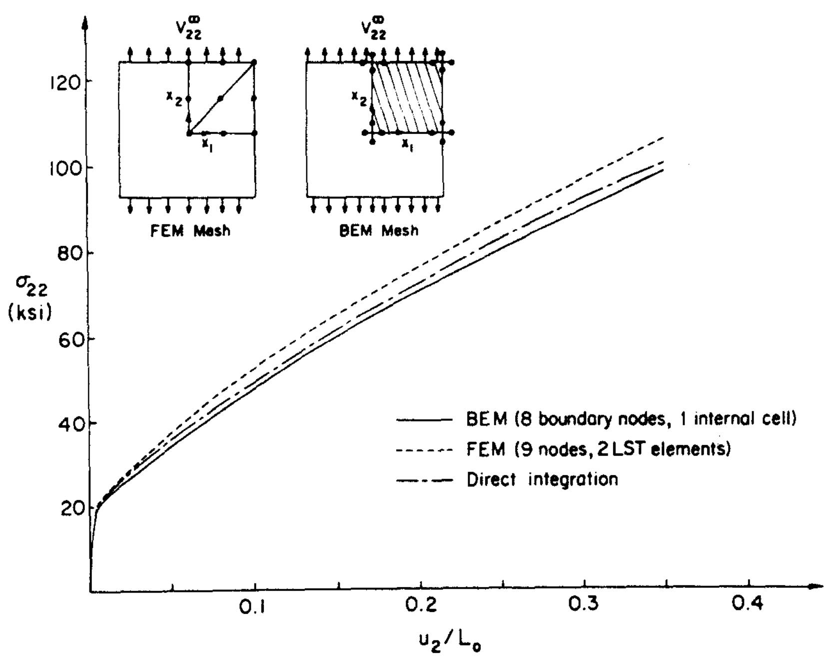

Fig. 3.

Table 3. BEM and FEM program statistics

Problem	Boundary Element Method			Finite Element Method		
	Boundary	Internal	CPU (sec)	Nodes	Elements	CPU (sec)
1. Uniaxial Extension (plane stress)	8	1	24.5	9	2	45
2. Uniaxial Extension (plane strain)	8	1	24.5	9	2	45

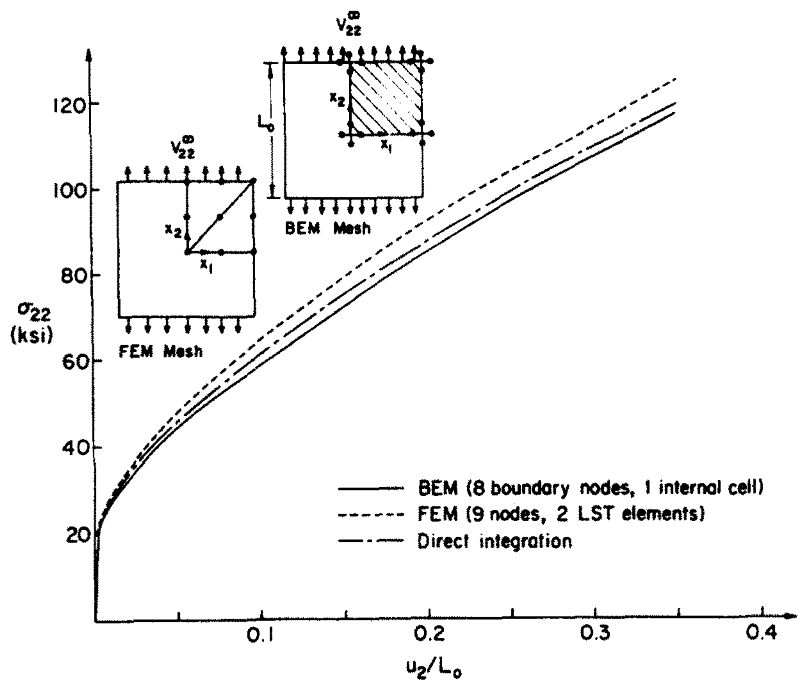

Fig. 4.

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